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A Random Surface Wave Equation for Bosonic QCD($SU(\infty)$)

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Abstract We propose a quantum surface wave functional describing the interaction between a colored $SU(N_c)$ membrane and a quantized Yang-Mills field. Additionally, we deduce its associated wave equation in the t'Hooft $N_c \to \infty$ limit. We show that its reproduces the Yang-Mills Field Theory at a large rigid random surface scale.

Keywords Non-perturbative QCD · String representation theory

1 Introduction

Still remains as the most important problem in quantum physics, the construction of a consistent mathematical framework in order to evaluate elementary particles scattering amplitudes and the correct understanding of the "chemistry" of the low energy nuclear quantum physics. The present candidate to describe the strong nuclear force is certainly bosonic QCD. However this mathematical framework has not been well understood from a correct calculational point of view. For instance, the description of higher energy scattering is entirely based on the assumption of the theory's coupling constant asymptotic freedom, a property that holds true only under the quite stringent condition on the flavor charge ($10 \le N_F \le 16$) with the low order loop-radiative corrections on the infrared regularized Feynman diagrammatics.

By the pure numerical calculational side, it should be remarked that Yang-Mills on lattice is a string theory on lattice, whose numerical predictions are strongly based on non-trivial fittings of non-perturbative parameters (QCD's low energy mesons masses parameters). Since QCD in a "stringy form" on lattice has been quite useful, it has lead A.M. Polyakov to propose describe QCD by $2D$-quantum gravity interacting with $2D$-massless fields as dynamical quantum fields of a string [1–3].

It is thus expected that the quantum amplitude of a closed trajectory of the quark in the vacuum of a pure Yang-Mills field should be given by the following path integral (after

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disregarding the classical quarks trajectory) [1–3]

$$
\begin{aligned}
Z &= \int D^{\text{cov}}[g_{ab}(\xi)] \int D^{\text{cov}}[X^\mu(\xi)] \\
&\quad \times \delta^{(F)}(g_{ab}(\xi) - e^{\varphi(\xi)}\delta_{ab}) \cdot e^{-u_0^2 \int \sqrt{g} d^2\xi} \\
&\quad \times \exp\left\{-\frac{1}{2\pi\alpha'} \int (\sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\mu)(\xi) d^2\xi\right\} \\
&\overset{(\text{after } 2\pi\alpha' = 1)}{=} \int \left[\prod_\xi d(e^{\varphi/2})(\xi)\right] \\
&\qquad \times \exp\left\{-\frac{(26-D)}{48\pi} \int \frac{1}{2}(\partial\varphi)^2 d^2\xi\right\} \\
&\qquad \times \exp\left\{-\left[u_0^2 + \lim_{\varepsilon \to 0^+}\left(\frac{2-D}{\varepsilon}\right)\right] \int (e^\varphi)(\xi) d^2\xi\right\}.
\end{aligned}
\tag{1a}
$$

It is worth point out that the Liouville $2D$-string theory in (1a) differs from that one considered earlier by A.M. Polyakov, although similar in its form

$$
Z = \int \left[\prod_\xi d(e^{\varphi/2})(\xi)\right] \exp\left\{-\frac{(26-D)}{48\pi} \int \left[\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}u_{\text{Poly}}^2 e^\varphi\right](\xi)\right\} \tag{1b}
$$

which means that the $2D$-cosmological constant term u_{Poly}^2 as given by A.M. Polyakov is given by

$$
u_{\text{Poly}}^2 = \frac{96\pi}{(26-D)}\left[u_0^2 + \lim_{\varepsilon \to 0^+}\left(\frac{2-D}{\varepsilon}\right)\right] \tag{1c}
$$

and leading to the anomaly on the total theory stress momentum-energy tensor which in principle does not vanish at $D = 26$

$$
\langle T_u^{u,\text{total}} \rangle = \frac{26-D}{48\pi}(R(\xi) + u_{\text{Poly}}^2). \tag{1d}
$$

At this point it is worth remark that if $u_0^2 \neq 0$, the decoupling of the quantum fluctuations of the ill-defined intrinsic Liouville string field $\varphi(\xi)$ in the path integral framework can be only accomplished through the formal saddle-point limit of $D \to -\infty$ in the resulting path integrals evaluations. Note that even in this Liouville quantum field decoupling, the effects of the non triviality of the covariant string is felt by means of its classical configurations which carry boundary conditions and non trivial topology on its mathematical structure. Note that only at $D \to -\infty$ one enforces the full preservation of the diffeomorphism invariance of the string path integral, even for those $2D$-coordinate change which acts as Weyl re-scalings of the intrinsic metric field $g_{ab}(\xi)$. It is useful to point out that covariant strings as a Liouville $2D$ quantum field appears to be problematic since one can always argue that by changing the non-linearity of the Liouville path measure (1b), to a linear Feynman measure, the string critical dimension decreases for $D = 25$! a result which is in principle in clear disagreement with the critical dimension obtained from the imposition of Lorentz Invariance on the operational framework.

Note that the $D \to -\infty$ saddle point limit is better understood in the following Liouville quantum path integral

$$Z(j) = \int \prod_{\xi} \left(e^{(\sqrt{\frac{48}{26-D}})\bar{\varphi}(\xi)}\right) \times \exp\left(-\frac{1}{2}\int_{\xi} (\partial\bar{\varphi})^2(\xi)\right) \times \exp\left(-\mu^2(\varepsilon)\int_{\xi} \exp\left(\sqrt{\frac{48}{26-D}}\bar{\varphi}\right)(\xi)\right) \times \exp\left(i\int_{\xi}\left(j\cdot\left(\frac{26-D}{48}\right)^{-\frac{1}{2}}\bar{\varphi}\right)(\xi)\right). \quad (1e)$$

One can see that $D \to -\infty$ means the "flow" of the above written non-linear Liouville path integral measure to classical configurations (with boundary terms).

It appears interesting at this point to call the readers attention that on light of the Nash Theorem on Riemannian Geometry, the Polyakov's path integral looses certainly its mathematical/geometrical meaning of a quantum string path integral since it is well-known that any intrinsic metric $g_{ab}(\xi)$ defined on the string world sheet can always be obtained from an immersion of the string world sheet into the ambient space R^D. So there is clearly an overcounting of would be independent degrees of freedom in the Polyakov's proposal! As a consequence one needs to enforces this constraint through the identity $\delta^{(F)}(g_{ab} - \frac{\partial x^{u}}{\partial \xi^{a}}\frac{\partial x_{u}}{\partial \xi^{b}})$ from the very beginning in a real string theory (see Luiz C.L. Botelho, P.RD 49, No. 4, 1979–1979 (1994)) and [3, Chap. 12].

However, in last decade, some sort of Polyakov's string moving on background has been considered as feasible candidates for defining QCD. The main point in this framework is the use of non linear sigma models in the perturbative scheme of D. Friedan (nonlinear sigma models in two minus epsilon dimensions, Phys. Rev. Lett. 45, 1057 (1980)). We feel that these string sigma model perturbative calculations when taken crudely outside condensed matter phenomenological quantum field theories has some unclear and ill-defined points. Let us highlight our doubts. In the D. Friedan framework, one considers the nonlinear sigma model interactions by regarding the external fields representing the sources for the string vertex excitations as fully coupling constants subject to the renormalization effects and with its value restricted by the condition that its associated β-function vanishes, so producing a perturbative conformal σ-model (with the vanishing of the trace of the momentum-energy $2D$-tensor) as the underlying string theory. Note that in order to implement the D. Friedan proposal in string theory, one utterly needs to introduce a two-dimensional infrared regularized mass term for the string vector position (which without this regularization is a ill-defined quantum field when considered in R^2). Fortunately this infrared cut-off disappears from the final (dimensional regularized) result for the β-function. It is additionally need to consider from the very beginning that the Liouville field is non propagating as a workable calculational assumption

$$[G_{\mu\nu}(X) = \delta_{\mu\nu} + \alpha' h_{\mu\nu}(X) + O(\alpha'^2);$$
$$X^{\mu}(\xi) = \overline{X}^{\mu} + \alpha' Y^{\mu}(\xi) + O(\alpha'^2),\ g_{ab}(\xi) = \delta_{ab},\ \text{etc.}\ldots].$$

The above mentioned steps on the D. Friedan conformal invariance σ-model proposal reflects the following impositions on the string path integral measures

$$\prod_{\xi}\left[\sqrt{G_{\mu\nu}(X^{\gamma}(\xi))}\cdot\sqrt[4]{g(\xi)}dX^{\mu}(\xi)\right]\sim\prod_{\xi}dX^{\mu}(\xi)+O(\alpha')(\alpha'), \tag{1f}$$

$$\prod_{\xi}\left[d(e^{\varphi/2})(\xi)\right]\overset{(\text{at } D=26)}{=}\prod_{\xi}d\varphi(\xi). \tag{1g}$$

The attractive point on the D. Friedan perturbative proposal for strings moving in backgrounds is that they satisfy the classical Einstein field equations (the classical Schwinger quantum field source in string theory is not fully arbitrary, but quite restricted and are classical field dynamical configurations) at the σ-model one-loop approximation in the Regge slope parameter α'. (C.G. Callan, E.J. Martinec, M.J. Perry and D. Friedan, Nucl. Phys. B 26, 593 (1985).)

In this paper, we take a different route by proposing an old idea [5–7, 11] to replace the one-dimensional closed trajectory in the above quantum amplitude by a two-dimensional random surface possessing color degrees as another collective non-perturbative effective variable for probing non-pertubative structures on QCD($SU(\infty)$). Thus, we deduce (formally) its associated surface wave equation in the t'Hooft topological limit of large number of colors $N_c=+\infty$. This study is presented in Sect. 2. On the Sect. 3 we suggest a path-integral argument on the connection of our proposed Random Surface Wave functional and the Path-Integral Partition Functional of the usual (Bosonic) Yang-Mills Gauge theory. Finally on Sect. 3 and Appendix B, we make some comments on previous work on the subject and on the regularization program.

2 The Random Surface Wave Functional

Let us start our analysis by considering the problem of associating a wave functional for a random surface Σ possessing $SU(N)$ color degrees of freedom interacting with an external quantized Yang-Mills field $A_{\mu}(X)$, the most simple geometrical gauge-invariant generalization of the usual Wilson Loop variable for QCD.

The colored random surface is characterized by two fields: first, by the usual (bosonic) vector position $X_{\mu}(\xi)$, $\xi\in D$ ($\mu=1,\dots,D$, where D is the space-time dimension), and second, by the random surface color variable $g(\xi)$ which is an element in the fundamental representation of the $SU(N)$ group [note that one should not consider theses string intrinsic degrees of freedom as new "compactified" dimensions]. Here, we have fixed the two-dimensional flat domain D to be the rectangle

$$D_{|0,2\pi|\times|0,T|}=\{(\xi_0,\xi_1);\ 0\le\xi_0\le 2\pi \text{ and } 0\le\xi_1\le T\}.$$

The classical action for this membrane is naturally given by [8–11].

$$S=S_0+S_1^{(B)} \tag{2}$$

with

$$S_0=\frac{1}{2}\int_D d^2\xi(\partial_a X^{\mu}\partial_a X^{\mu})(\xi), \tag{2a}$$

$$S_1^{(B)} = \frac{1}{4\pi m} \int_D T_R^{(c)} (g^{-1}\partial_a g)^2(\xi) d^2\xi + 4\pi i \Gamma_{\mathrm{WZ}[g]}, \tag{2b}$$

where $\Gamma_{\mathrm{WZ}[g]}$ denotes the two-dimensional Wess-Zumino functional. Its existence, together with the integer m in the above written σ-model on the action of $g(\xi)$'s afford us to consider the bosonized fermionic equivalent action

$$S_1^{(F)} = \int_D \psi(\xi)(i\gamma_a\partial_a)\overline{\psi}(\xi)d^2\xi, \tag{3}$$

where the two-dimensional Dirac field $\psi(\xi)$ belongs to the fermionic fundamental $SU(N)$ representation.

At this point, the simplest action taking into account the interaction with the external non-Abelian field is given by

$$S^{\mathrm{int}}[\psi(\xi); A_\mu(X)] = \int_D \overline{\psi}(\xi)(\gamma_a\partial^a X^\mu(\xi)A_\mu(X(\xi))\psi(\xi)d^2\xi. \tag{4}$$

The complete classical interacting action ((2), (3) and (4)) is invariant under the gauge transformations

$$\begin{aligned} A_\mu(X_\mu(\xi)) &\to (h^{-1}A_\mu h + h^{-1}\partial_\mu h)(X_\mu(\xi)), \\ \psi(\xi) &\to h(X_\mu(\xi))\psi(\xi), \\ \overline{\psi}(\xi) &\to \overline{\psi}(\xi)h^{-1}(X_\mu(\xi)). \end{aligned} \tag{5}$$

Before turning to the construction of a quantum wave functional for the above system, it is instructive to remark that (2), (3) and (4) are the random surface generalizations of the analogous formulae in the one-dimensional string case, where the colored string is described by the position vector $X_\mu(\sigma)$ and the one-dimensional complex fermion (Grassmanian) field $\{\theta(\sigma), \theta^\star(\sigma)\}$ in the $SU(N)$ fundamental representation. The associated action is

$$\begin{aligned} &S[X_\mu(\sigma), \theta(\sigma), \theta^\star(\sigma), A_\mu(X_\mu(\sigma))] \\ &\quad = \int_0^T \frac{1}{2}\dot{X}_\mu(\sigma)^2 d\sigma + \int_0^T \theta^\star(\sigma)\dot{\theta}(\sigma) + \int_0^T \dot{X}^\mu(\sigma)A_\mu^I(X(\sigma))(\theta(\sigma)\lambda_I\theta^\star(\sigma))d\sigma, \end{aligned} \tag{6}$$

where $\{\lambda_I\}$ denotes the Hermitian generators of the $SU(N)$ Lie algebra.

In this string case, a quantum wave functional is given by the following path integral (2)

$$\begin{aligned} &W[X_\mu(\sigma), A_\mu(X)] \\ &\quad = \int d[\theta(\sigma)]d[\theta^\star(\sigma)] \sum_{\alpha=1}^{N^2-1} \theta_\alpha(0)\theta_\alpha^*(T)\exp\{-S[X_\mu(\sigma), \theta(\sigma), \theta^\star(\sigma), A_\mu(X_\mu(\sigma))]\} \end{aligned} \tag{7}$$

which leads to the well-known Wilson Loop factor defined by the closed string $\{X_\mu(\sigma)\}$. The complete quantum wave functional is defined by the average $\langle W[X_\mu(\sigma), A_\mu(X)]\rangle$ where $\langle\ \rangle$ denotes the partition functional of the pure Yang-Mills theory [1–3].

We shall now use (7) to propose the following functional integral as a quantum wave functional for a $SU(N)$ colored random surface Σ interacting with the quantum vacuum of a $SU(N)$ Yang-Mills theory.

$$\mathrm{Tr}^{\mathrm{color}}(\psi[\Sigma]) \stackrel{\mathrm{def}}{=} \sum_{R=1}^{N^2-1} \int d[\psi(\xi)]d[\overline{\psi}(\xi)]\left(\overline{\psi}(0,0)\frac{\lambda^R}{N_c}\psi(2\pi,0)\right) \times \exp\{-S[X_\mu(\xi), A_\mu(X(\xi),\psi(\xi))]\}. \tag{8}$$

Notice that our above proposed random surface phase factor $\mathrm{Tr}^{\mathrm{color}}(\psi[\Sigma])$ is a 2×2 matrix in the flat domain $D(a=1,2)$.

In order to deduce a closed wave functional for the quantum average $\langle \mathrm{Tr}^{\mathrm{color}}(\psi[\Sigma])\rangle$ in the limit $N_c = +\infty$, we proceed as in the string case (2, 3) by shifting the $A_\mu(X)$ field variable, which by its turn, produces the following result ($\lambda_0^2 = \lim_{N^c\to\infty}(g_0^2 N_c) < \infty$)

$$\frac{1}{4\lambda_0^2}\langle \mathrm{Tr}^{\mathrm{color}}\{(D_\mu F_{\mu\nu})(X)\psi[\Sigma]\}\rangle = \int_D \delta^{(D)}(X - X_\mu(\sigma,\xi))\partial_c X^\mu(\sigma,\tau)\langle \mathrm{Tr}^{\mathrm{color}}\psi[\Sigma_1]\rangle\langle \mathrm{Tr}^{\mathrm{color}}\gamma^{(c)}\psi[\Sigma_2]\rangle, \tag{9}$$

where the split membranes $\Sigma_{(1)}$ and $\Sigma_{(2)}$ are respectively defined by the restriction of the mapping $X_\mu(\xi_1,\xi_2)$ for the (split) domains

$$D_{(1)} = \{(\xi_0,\xi_1);\ 0\le \xi_0\le\sigma;\ 0\le\xi_1\le T\}$$

and

$$D_{(2)} = \{(\xi_0,\xi_1)\mid \sigma\le\xi_0\le 2\pi;\ 0\le\xi_1\le T\}.$$

It is now convenient to multiply both sides of (9) by the membrane current density

$$J_{(a)}(X) = \delta^{(D)}(X - X_\mu(\overline{\sigma},\overline{\tau}))\partial_a X^\mu(\overline{\sigma},\overline{\tau})$$

and integrate out the result relative to the space-time variable X. So, we get the result

$$\langle \mathrm{Tr}^{\mathrm{color}}\{(D_\mu F_{\mu\nu})(X^\mu(\overline{\sigma},\overline{\tau}))\partial_a X^\mu(\overline{\sigma},\overline{\tau})\psi[\Sigma]\}\rangle = 4\lambda_0^2\int_D \delta^{(D)}(X_\mu(\overline{\sigma},\overline{\tau}) - X_\mu(\sigma,\tau))\partial_c X^\mu(\sigma,\tau)\partial_a X^\mu(\overline{\sigma},\tau) \times \langle \mathrm{Tr}^{\mathrm{color}}\psi[\Sigma_{(1)}]\rangle\gamma^{(c)}\langle \mathrm{Tr}^{\mathrm{color}}\psi[\Sigma_{(2)}]\rangle. \tag{10}$$

In order to write the left-hand side of the above result in a form similar to the random surface wave equation of [15, 16] we use the relations

$$\left\{\frac{\delta}{\delta X_\mu(\sigma,\tau)}\right\}\mathrm{Tr}^{\mathrm{color}}(\psi(\Sigma)) = \mathrm{Tr}^{\mathrm{color}}(\psi(\Sigma_1)F_{\mu\nu}(X(\sigma,\tau))\partial_c X^\nu(\sigma,\tau)\gamma^{(c)}\psi(\Sigma_2)), \tag{11a}$$

$$P_F\left\{\frac{\delta^2}{\delta X_\mu(\overline{\sigma},\overline{\tau})\delta X^\mu(\sigma,\tau)}\right\}\mathrm{Tr}^{\mathrm{color}}(\psi(\Sigma)) = \mathrm{Tr}^{\mathrm{color}}(D_\mu F_{\mu\nu}(X_\mu(\overline{\sigma},\overline{\tau}))\partial_c X^\nu(\sigma,\tau)\gamma^{(c)}\psi(\Sigma)), \tag{11b}$$

where the derivative-finite part operations is given by [1–3].

$$P_F\left\{\frac{\delta^2}{\delta X_\mu(\overline{\sigma},\overline{\tau})\delta X^\mu(\overline{\sigma},\overline{\tau})}\right\}$$
$$\equiv \lim_{\varepsilon\to 0^+}\int_{-\varepsilon}^{\varepsilon} d\beta\,\frac{\delta^2}{\delta X_\mu(\overline{\sigma}+\beta,\overline{\tau}+\beta)\delta X^\mu(\overline{\sigma}-\beta,\overline{\tau}-\beta)}. \tag{11c}$$

By substituting (11b) into (10), we obtain our proposed random surface version of the string Polyakov-Migdal-Makkenko wave equation

$$P_F\left\{\frac{\delta^2}{\delta X_\mu(\overline{\sigma},\overline{\tau})\delta X^\mu(\overline{\sigma},\overline{\tau})}\right\}\langle \mathrm{Tr}^{\mathrm{color}}(\psi(\Sigma))\rangle$$
$$=4\lambda_0^2\int_D \delta^{(D)}(X_\mu(\overline{\sigma},\overline{\tau})-X_\mu(\sigma,\tau))\partial_b X^\mu(\sigma,\tau)\partial_c X^\mu(\overline{\sigma},\overline{\tau})$$
$$\times\langle \mathrm{Tr}^{\mathrm{color}}\gamma^{(b)}\psi[\Sigma_{(1)}]\rangle\langle \mathrm{Tr}^{\mathrm{color}}\gamma^{(c)}\psi[\Sigma_{(2)}]\rangle. \tag{12a}$$

To summarize, we propose a continuum random surface version of the string Migdal-Makkenko loop wave equation in $SU(\infty)$, which we hope to open a new path to understand the non-pertubative structure of Quantum Chromodynamics as a dynamics of random surfaces as much successful studies implemented in Loop Space approach for Quantum Gravity [15, 16].

3 A Connection with QCD($SU(\infty)$)

In this section we present an path-integral argument connecting our proposed random surface wave functional (8) to the QCD($SU(\infty)$), thus, showing the usefulness of our propose on Sect. 2.

In order to achieve such goal, let us consider the quantum vacuum of the Yang-Mills theory as an ensemble of random $SU(N)$ connections with *an uniform distribution* interacting with the random surface Σ constraint to remains on the sphere S^{D-1} on R^D. Formally one is considering the strong bare coupling $g^2_{\mathrm{bare}}\to\infty$ vacuum limit on the Yang-Mills quantum average and the random surface rigid limit $X_\mu(\xi)=\overline{X}_\mu+\sqrt{\alpha'}Y_\mu(\xi)$, with $\alpha'\to 0$ denoting the physical observable Regge slope constant, namely:

$$\langle \mathrm{Tr}^{\mathrm{color}}(\psi(\Sigma))\rangle_{g^2\to\infty}=\int(\Pi_{(\overline{X},\mu,a)}dA^a_\mu(\overline{X}))^{\mathrm{Haar}}\int D^F[X^\mu(\xi)]$$
$$\times\exp\left[-\frac{1}{2}\int_{R^2}d^2\xi(\partial_A X^\mu\partial^A X_\mu)(\xi)\right]\delta^{(F)}((X_\mu X^\mu(\xi)-1)$$
$$\times\int D^F[\psi_a,\overline{\psi}_a]\exp\left[-\frac{1}{2}\int_{R^2}d^2\xi(\overline{\psi}(i\gamma^A\partial_A)\psi)(\xi)\right]$$
$$\times\exp\left[ie\int_{H^2}d^2\xi\left[A^i_\mu(X^\beta(\xi))(\overline{\psi}_a\gamma^A(\lambda_i)_{ab}\psi_b)(\xi)(\partial_A X^\mu)(\xi)\right]\right]. \tag{12b}$$

In order to connect (12b) with QCD($SU(\infty)$), we consider the "Harmonic gauge" fixing in the Haar-Yang-Mills path integral in (12b), namely $(X_\mu(\xi)-\overline{X}_\mu)\cdot A_\mu(X^\beta(\xi))=0$, which allow us in its turn to rewrite the interaction term in (12b) in terms of the Yang-Mills strength field in the chart $V(\overline{X})$ at large random surface scale ($\alpha' \to 0$), since in this harmonic gauge we have the expansion $A_\mu(X^\beta(\xi)) = -\frac{1}{2}F_{\mu\nu}(\overline{X}^\alpha)\sqrt{\alpha'}Y^\nu(\xi) + O(\sqrt{\alpha'})$

$$\begin{aligned} I_{V(\overline{X})}[A_a^\mu(\overline{X})] &= \int D^F[Y^\mu(\xi)]\exp\left[-\frac{1}{2}\int_{R^2} d^2\xi(\partial_A Y^\mu \partial^A Y_\mu)(\xi)\right] \\ &\quad \times \left(\lim_{\lambda\to\infty}\exp\left[-\langle\lambda\rangle\int_{R^2} d^2\xi[(Y^\mu Y_\mu)(\xi) - 1]\right]\right) \\ &\quad \times \int D^F[\psi_a, \overline{\psi}_a]\exp\left[-\frac{1}{2}\int_{R^2} d^2\xi(\overline{\psi}(i\gamma^A\partial_A)\psi)(\xi)\right] \\ &\quad \times \exp\left[-ie\alpha'\int_{R^2} d^2\xi\frac{1}{2}Y^P(\xi)F^i_{\rho\mu}(\overline{X}^\alpha)(\overline{\psi}_a\gamma^A(\lambda_i)_{ab}\psi)(\xi)(\partial_A Y^\mu)(\xi)\right]. \end{aligned} \tag{13}$$

Note that we have used the condensate Polyakov approximation [1–3] for the functional delta inside (12), expected to hold true in the limit of $\alpha' \to 0$ and effectively generating a mass term for the random surface vector position field

$$\begin{aligned} &\delta^{(F)}((X^\mu X_\mu)(\xi) - 1) \\ &\quad = \int D^F[\lambda(\xi)]e^{+i\int_{R^2} d^2\xi\lambda(\xi)[(X^\mu X_\mu)(\xi)-1]} \\ &\quad \sim \lim_{\langle\lambda\rangle\to\infty}\left\{e^{+i\int_{R^2} i\langle\lambda\rangle_{\text{conden}}[(X^\mu X_\mu)(\xi)-1]d^2\xi}\right\} \\ &\quad \sim \lim_{\langle\lambda\rangle\to\infty} e^{-\langle\lambda\rangle_{\text{conden}}\int_{R^2} d^2\xi[(X^\mu X_\mu)(\xi)]}. \end{aligned} \tag{14a}$$

At this point we evaluate the $Y_\mu(\xi)$-Gaussian functional integral with the exact result

$$\begin{aligned} &I_{V(\overline{X})}[A_a^\mu(\overline{X})] \\ &\quad = \lim_{\langle\lambda\rangle\to\infty}\left\langle \det{}^{-\frac{1}{2}}\left[(-\partial^2)_\xi\eta^{\mu\nu}(\overline{X}) + \frac{1}{2}(F_i^{\mu\nu}(\overline{X})j_a^i(\xi))\partial_\xi^a + \langle\lambda\rangle\right]\right\rangle_{\psi,\overline{\psi}}, \end{aligned} \tag{14b}$$

where $\langle,\rangle_{\psi,\overline{\psi}}$ denotes the functional integral over the $SU(N)$ string intrinsic Dirac fields and $j_a^i(\xi)$ is the conserved fermion $SU(N)$ current on the random surface sheet.

At the condensate value $\langle\lambda\rangle \to \infty$, we obtain the following result for (14)

$$I_{V(\overline{X})}[A_\mu^a(\overline{X})] \sim \left\langle \exp\left[-\frac{1}{16\pi}F^i_{\mu\nu}(\overline{X})F^\rho_{\mu\nu}(\overline{X})(j_i^a(\xi)j_\rho^a(\xi))\right]\right\rangle_{\psi,\overline{\psi}} \tag{15}$$

which at large N, give us the final result depending only the "infinite-tensioned random surface macroscopic space-time fixed vector position $\overline{X}$"

$$I_{V(\overline{X})}[A_\mu^a(\overline{X})]_{(N\to\infty)} = \exp\left[-\left(\frac{\langle\int d^2\xi j_i^a(\xi)j_i^a(\xi)\rangle_{\psi,\overline{\psi}}^{(N\to\infty)}}{16\pi}\right)F^i_{\mu\nu}(\overline{X})F^i_{\mu\nu}(\overline{X})\right]. \tag{16}$$

The complete path integral equation (16) is, thus, exactly the $SU(\infty)$ Yang-Mills quantum field path integral for the space-time at large random surface scale (after integrating out the space-time microscopic surface space-time point $\overline{X}$)

$$\int D^F[A_\mu^a(\overline{X})]\exp\left[-\frac{1}{16g_{\rm QCD}^2}\int d^D\overline{X}(F_{\mu\nu}^2(\overline{X}))\right]$$
$$= \sum_{\text{(membranes)}} \left\{\left\langle \mathrm{Tr}^{\rm color}(\psi(\Sigma))\right\rangle_{g_{\rm bare}^2\to 0}^{N_c\to\infty}\right\}. \qquad (17)$$

Note that the $\mathrm{QCD}_{N_c=+\infty}$ coupling constant is expressed in terms of the intrinsic Fermion fields in an explicitly form

$$(g_{\rm QCD})^{-2} = \frac{1}{\pi}\left\langle \int d^2\xi\, j_i^a(\xi)\, j_i^a(\xi)\right\rangle_{\psi,\overline{\psi}}^{(N\to\infty)}. \qquad (18)$$

Appendix A: Rank Two Antisymmetric Path-Integrals and the QCD String: Some Comments

The most important problem in the present days of theoretical and mathematical physics is how to quantize correctly Non-Abelian Gauge Field Theories defined on the physical continuum space-time. The only result in this direction still remains a somewhat formal Ansatz from the experimental and theoretical point of view of the use of the Higgs mechanism. Probably, this Ansatz is formal from a strict quantum field theoretic point of view since its makes heavier use of a trivial $\lambda\phi^4$-field theory in four dimensions and of the associated gauges of t'Hooft for the Yang-Mills Fields (see the comments on p. 38 the J.C. Taylor book "Gauge theories of weak interactions". Cambridge Monographs on Mathematical Physics). However, it was realized by K. Wilson that in the Ising like euclidean path integral crude approximation framework (Lattice Gauge Theory) theses non-Abelian gauge field theories in the lattice at a bare strong coupling regime are naturally expressed in terms of the *Euclidean* Wilson Loops defined by the matter content trajectories $C = \{X_\mu(\sigma); 0\le\sigma\le 1\ \sigma = \text{proper-time parameters}\}$

$$W[C] = \mathrm{Tr}\,\mathbb{P}\left\{\exp\left[+i\oint_C A_\mu dX^\mu\right]\right\}. \qquad (A.1)$$

Note that typical interaction energy densities, such as $\overline{\psi}\psi$, $\overline{\psi}\gamma^5\psi$, $\overline{\psi}\gamma^\mu\psi A_\mu$ which are real function (distributions) in the Minkowski space-time are complex on the Euclidean world.

It was argued on [14] by A.M. Polyakov, an euclidean string functional integral Ansatz for (A.1) based on a coupling of an Abelian rank-two antisymmetric tensor field $B_{\mu\nu}(x)$ (the Polyakov's axion field) with the string orientation area tensor previously proposed by this author but with an important difference: This rank-two antisymmetric tensor field B has a non trivial dynamic content. Namely (see (13)–(16)), ref. [14]).

$$W[C] = \frac{\int D^F[B_{\mu\nu}]\, e^{-S[B_{\mu\nu}]}\, e^{(i\int_{\Sigma_C} B d\sigma)}}{\int D^F[B_{\mu\nu}]\, e^{-S[B_{\mu\nu}]}}, \qquad (A.2)$$

where the axion action is given by

$$S(B) = \frac{1}{4e^2}\int d^\nu x\left(B_{\mu\nu}^2 + dB\cdot \text{arc sen}\frac{dB}{m^2} - \sqrt{m^4-(dB)^2}\right). \qquad (A.3)$$

At this point we point out that the functional integral weight (A.3) makes sense only for those field configurations which makes (A.3) a *real number*, namely: $\sup_{x\in R^\nu}|dB(x)| \le m^2$.

Unfortunately this bound on the kinetic energy of the axion field is impossible for those distributional fields configurations making the domain of the axion functional integral equation (A.2), unless $m^2 \to \infty$ and comments below (40) of ref. [14]. (A quantum field may be bounded but not its kinetic energy!)

So, in the deep infrared regime of QCD($SU(\infty)$) (A.3) should turns into a pure White-Gaussian action for the axion field B dominated by almost constant gauge field configurations

$$S[B] \sim \frac{1}{4e^2}\int B^2(x)d^\nu x. \tag{A.4}$$

One has, thus, the following effective result for the Wilson loop surface dependence in the very low momenta regime

$$W[C] \sim \exp\left[-F\left(C;\sum_C\right)\right], \tag{A.5}$$

where the surface functional weight is given by the self-avoiding extrinsic action firstly proposed in a minimal area context solution for the QCD-Loop wave equation in ref. [1–3] with β a (positive) coupling constant

$$F\left(C,\sum_C\right) = \beta\int_\Sigma d\sigma_{\mu\alpha}(x)(\delta^{\mu\lambda}\delta^{\alpha\beta} - \delta^{\mu\beta}\delta^{\lambda\rho}(x-y))d\sigma_{\lambda\rho}(y). \tag{A.6}$$

It is straightforward to see that for fixed constant e^2, the limit $m^2 \to \infty$ leads to a pure Nambu-Goto action strongly coupled [14]

$$\begin{aligned} F\left(C,\sum_C\right) &\sim \lim_{m^2\to\infty} c_1(e^2 m)\int d^2\xi\sqrt{g}(\xi) + \lim_{m^2\to\infty} c_2(e^2/m)\int d^2\xi(\nabla t_{\mu\nu})^2\sqrt{g} \\ &\quad + O\left(\frac{1}{m}\right) \sim \frac{1}{2\pi\alpha'}\int d^2\xi\sqrt{g}(\xi) \end{aligned} \tag{A.7}$$

and by its turn suggesting a random surface wave functional behavior like (8) for the quantum averaged QCD($SU(\infty)$) Wilson loop (7)–(A.1) in the QCD($SU(\infty)$) deep infrared regime.

Appendix B: On the Self-Avoiding Membrane Wave Functional

In this appendix we present some comments on the renormalization program to the random surface wave functional associated to the self-avoiding extrinsic reparametrization functional for QCD($SU(\infty)$) in R^D as given by (A.5) of the previous Appendix A

$$\begin{aligned} F[X^\alpha(\xi)] &= \beta\int_\xi\sqrt{h}(X(\xi))\int_\xi\sqrt{h(X(\xi'))}(\Gamma^{\mu\nu}(X(\xi))\Gamma_{\mu\nu}(X(\xi')) - 1) \\ &\quad\times\delta^{(D)}(X^\alpha(\xi) - X^\alpha(\xi')). \end{aligned} \tag{B.1}$$

With $X^\alpha(\xi)$ denoting the parametrization of the Σ-surface on the surface wave function ansatz (A.5).

Here the surface area tensor responsible by the extrinsic properties of QCD($SU(\infty)$) quantum geometry is explicitly given by

$$\Gamma^{\mu\nu}(X(\xi)) = \frac{(\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu)_{(\xi)}}{\sqrt{h(X(\xi))}} \tag{B.2}$$

and the random surface scalar area is written as

$$\sqrt{h(X(\xi))} = \sqrt{\det\{\partial_a X^\mu \partial_b X_\mu\}(\xi)}. \tag{B.3}$$

As a first step to analyze (B.1), one should extract the pure string world sheet UV divergence associated to the trivial self-avoiding surface case $X_\mu(\xi) = X_\mu(\xi')$ with $\xi = \xi'$.

Let is follow our study.

Firstly we note that a regularized form for (B.1) in the UV case $\xi = \xi'$ is explicitly given by

$$W_{(\Lambda)}[X(\xi)] = \beta \sum_{p=0}^{\infty} \frac{(-1)^p}{p!2^{2p}\cdot\Gamma(\frac{D}{2}+p)}\left(\frac{\Lambda^{D+2p}}{D+2p}\right)\cdot\delta_\Lambda(\xi,\xi')$$
$$\times X\left\{\int_\xi d_\xi^2 d_{\xi'}^2 \sqrt{h(\xi)}\sqrt{h(\xi')}(\Gamma^{\mu\nu}(X(\xi)\Gamma_{\mu\nu}(X(\xi')) - 1)|X(\xi) - X(\xi')|^{2p}\right\} \tag{B.4}$$

with

$$\delta_\Lambda(\xi,\xi') = \begin{cases} \Lambda & \text{if } \begin{cases} \xi_1 - \frac{1}{\Lambda} \le \xi_1' \le \xi_1 + \frac{1}{\Lambda}, \\ \xi_2 - \frac{1}{\Lambda} \le \xi_2' \le \xi_2 + \frac{1}{\Lambda}, \end{cases} \\ 0 & \text{otherwise.} \end{cases} \tag{B.5}$$

By considering the Taylor expansion around $\xi = \xi'$

$$\Gamma^{\mu\nu}(X(\xi))\Gamma_{\mu\nu}(X(\xi')) - 1 = -(\partial_a\Gamma^{\mu\nu})(\partial_b\Gamma_{\mu\nu}(X(\xi))(\xi-\xi')_a(\xi-\xi')_b + \text{higher terms} \tag{B.6}$$

one can see that all reparametrization invariant counter-terms are of the second order derivative on the surface vector position and on the area tensor object namely, at one-loop case ($p \le 1$); one has the following explicit counter-terms involving the extrinsic geometry (note the subtraction of the pure self-avoiding term in (2) which at the level of loop equations means that a non-vanishing Gluon condensate was already taken into account by considering a non-zero Regge slope parameter, i.e., $(2\pi\alpha')^{-1} = \langle 0|F^2|0\rangle \neq 0$:

$$\begin{aligned} W_1[X(\xi)] &\sim \beta(\Lambda)^4 \int_\xi \sqrt{h(\xi)}(\partial_a\Gamma^{\mu\nu}\partial^a\Gamma_{\mu\nu})(\xi), \\ W_2[X(\xi)] &\sim \beta(\Lambda)^4 \int_\xi \sqrt{h(\xi)}\{(\partial_a\Gamma^{\alpha\beta})(\partial^a\Gamma^{\mu\beta})(\partial_b X_\alpha)(\partial^b X_\mu) + \cdots\}, \\ W_3[X(\xi)] &\sim \beta\int_\xi \sqrt{h(\xi)}\{(\partial^2\Gamma^{\alpha\beta}(\partial^2\Gamma_{\alpha\beta})\}. \end{aligned} \tag{B.7}$$

At this point we consider the extrinsic ultraviolet divergences $X_\mu(\xi) = X_\mu(\xi')$ *but with* $\xi \neq \xi'$.

In the physical situation of line self-intersections, where the equation $X_\mu(\xi) = X_\mu(\xi')$ defines a sub-manifold of dimension 1 (the Σ-surface is generically described by the union of vertical surfaces cylinders locally in contact along self-intersecting vertical lines passing through the points $\sigma_j = \{\xi_j^1, \tau\}$ with $X_\mu(\xi_j', \tau) = X_\mu(\xi_{j+1}^1, \tau)$, $1 \le j \le m$). The resulting random surface wave functional path integral still remains formally renormalizable. In order to show the correctness of this claim, one can see that $\Gamma_{\mu\nu}(X(\sigma_j))\Gamma^{\mu\nu}(X(\sigma_{j+1})) = \cos X(\sigma_j; \sigma_{j+1})$, the constant angle between the extrinsic surface tangent planes possessing the common self-intersecting non-trivial line $X_\mu(\sigma_j)$ (or $X_\mu(\sigma_{j+1})$!). Now it is straightforward to see that the action (B.1) reduces to a pure (intrinsic) self-avoiding action of the cylinder surfaces branches with the associated tangent plane above cited. In this simple case one can follow our previous exposed results in [5, 6] to show its formal renormalizability as a two-dimensional Quantum Field Theory.

Appendix C: The Electro-Weak Sector as Dynamics of Loops and Strings

Let us consider a set o microscopical quark-leptions Fermi fields interacting with a color Yang-Mills field $A_\mu^\ell \lambda_\ell$ and a flavor Yang-Mills field $W_\mu^i \tau_i$

$$
\begin{aligned}
Z = \int & \mathcal{D}^F[\psi(x)]\mathcal{D}^F[\bar{\psi}(x)]\mathcal{D}^F[e(x)]\mathcal{D}^F[\bar{e}(x)]\mathcal{D}^F[A_\mu(x)]\mathcal{D}^F[W_\mu(x)] \\
& \times \exp\left\{-\frac{1}{4g^2}\int d^\nu x \mathrm{Tr}_{\mathrm{color}}(F_{\mu\nu}^2(A))(x) - \frac{1}{4e^2}\int d^\nu x \mathrm{Tr}_{\mathrm{flavor}}(F_{\mu\nu}^2(W))(x)\right\} \\
& \times \exp\left\{-\frac{1}{2}\int d^\nu x(\bar{\psi}\, \not{D}(A \otimes W)\psi)(x)\right\} \\
& \times \exp\left\{-\frac{1}{2}\int d^\nu x(\bar{e}\, \not{D}(W)e)(x)\right\} \\
& \times \exp\left\{-\frac{1}{2}\int d^\nu x(\bar{\eta}e + \bar{e}\eta)(x)\right\},
\end{aligned} \tag{C.1}
$$

where $\{\psi_{(x)}, \bar{\psi}\}$; $\{e(x), \bar{e}(x)\}$ denotes the quark and lepton fields respectively.

After integrating out the quark-lepton fields and by writing the effective action in Loop Space, we obtain the following expression

$$
\begin{aligned}
Z = \int & \mathcal{D}^F[A_\mu(x)]\mathcal{D}^F[W_\mu(x)] \\
& \times \exp\left\{-\frac{1}{4g^2}\int d^\nu x \mathrm{Tr}_c(F_{\mu\nu}^2(A))(x) - \frac{1}{4e^2}\int d^\nu x \mathrm{Tr}_F(F_{\mu\nu}^2(W))(x)\right\} \\
& \times \exp\left\{-\sum_{\{c_\mu(\sigma)\}}(\Psi[c_\mu(\sigma), 0 \le \sigma \le s](A_\mu)\Psi[c_\mu(\sigma), 0 \le \sigma \le s](W_\mu)\right\} \\
& \times \exp\left\{-\sum_{\{L_\mu \bar{\sigma})\}}\Psi[L_\mu(\bar{\sigma}), 0 \le \bar{\sigma} \le \bar{s}](W_\mu)\right\}
\end{aligned}
$$

$$\times \exp\left\{-\sum_{\{\bar{L}_\mu(s)\}}^{(xy)} \int d^\nu x d^\nu y \eta_\alpha(x) \psi^{(ab)}_{(xy)}[\bar{L}_\mu(\bar{\sigma}), 0 \le \bar{\sigma} \le \bar{s}](W_\mu)\bar{\eta}_b(\delta)\right\}. \quad (C.2)$$

Here, we have kept the subscript (A_μ) or (W_μ) to remind the reader about the gauge field kind entering in the definition of the Mandelstam-Feynman Phase Factors, namely

$$\Psi[c_\mu(\sigma), 0 \le \sigma \le s](A_\mu) = \frac{1}{N_c} \mathrm{Tr}_{\mathrm{color}} \mathbb{P}\left\{\exp\left(-i \int_0^s d\sigma A_\mu(c^\mu(\sigma))\dot{c}_\mu\right)\right\}, \quad (C.3)$$

$$\psi^{ab}_{(xy)}[\bar{L}_\mu(\sigma), 0 \le \sigma \le \bar{s}](W_\mu) = \mathbb{P}\left\{\exp\left(-i \int_0^s W_\mu(\tilde{L}_\mu(\bar{\sigma}))\dot{\tilde{L}}_\mu(\bar{\sigma})\right)\right\}. \quad (C.4)$$

The Bosonized sum over the *closed fermionic quark trajectories* are given by the following path integral which comes from the procedure of writing fermionic functional determinants in Loop Space

$$\sum_{\{c_\mu(\sigma)\}} \equiv -\int_0^\infty \frac{ds}{s} \int d^\nu x \int_{c_\mu(0)=c_\mu(s)=x_\mu} \mathcal{D}^F[c_\mu(\sigma)]\mathcal{D}^F(\pi_\mu(\sigma)]$$

$$\times \exp\left(i \int_0^s d\sigma(\pi^\mu(\sigma)\dot{c}_\mu(\sigma))\mathbb{P}_{\mathrm{Dirac}}\left[\exp\left(i \int_0^s d\sigma(\pi^\mu(\sigma)\gamma_\mu)\right)\right]\right). \quad (C.5)$$

In relation to the Bosonized sum over the *open Fermionic Lepton Feynman trajectories* which enters into the definition of the Lepton Propagator, it differs slightly for (C.5) and is given by

$$\sum_{(\tilde{L}_\mu(\sigma))} \equiv \int_0^\infty d\bar{s} \int_{\tilde{L}_\mu(0)=x_\mu; \tilde{L}_\mu(\bar{s})=y_\mu} \mathcal{D}^F[L_\mu(\sigma)]\mathcal{D}^F[\pi_\mu(\sigma)]$$

$$\times \exp\left(i \int_0^{\bar{s}} d\sigma(\pi^\mu(\sigma) \cdot \dot{\tilde{L}}_\mu(\sigma))\right)\mathbb{P}_{\mathrm{Dirac}} 9 \exp\left(i \int_0^{\bar{s}} d\sigma(\pi^\mu(\sigma) \cdot \gamma_\mu(\sigma))\right). \quad (C.6)$$

By using now the result (8) to evaluate exactly the Gluon Functional Integral in terms of a random surface theory in (C.2) we get our main result, which express the Strong-Electro-Weak quantum Field theory (C.1) in terms of a Dynamics of Interacting Contours

$$\{c_\mu(\sigma), 0 \le \sigma \le s\}; \quad \{L_\mu(\bar{\sigma}), 0 \le \bar{\sigma} \le \bar{s}\}; \quad \{\tilde{L}_\mu(\bar{\sigma}), 0 \le \sigma \le s\}$$

and our proposed random surface theory (8) for all topological sectors added with appropriate vertex for Mesons and Baryons Excitations.

The fundamental point on the String formulation for QCD is related to interaction with the Electro-Weak sector flavor charges. In this sector one postulate that the vertex associated to the string theory (8) is related to the Physical Baryon-Meson sector and at the point *one identify the intrinsic Generalized Elfin Charges wit flavor physical charges*. For instance, the scalar and vectorial flavored mesons are generated respectively by the vertex

$$V_{\mathrm{scalar}} = \exp\left\{-\int_{-\infty}^{+\infty} d\tau \int_0^{2\pi} d\sigma \sqrt{h(\sigma,\tau)} J_{(i)}(X(\sigma,\tau))(\bar{\psi}_{(k)}(\tau^i)_{(k)(p)}\psi_{(p)})(\sigma,\tau)\right\}, \quad (C.7)$$

$$V_{\text{scalar}} = \exp\left\{ -\int_{-\infty}^{+\infty} d\tau \int_0^{2\pi} d\sigma \sqrt{h(\sigma,\tau)} A_\mu^i(X(\sigma,\tau)) \right.$$
$$\left. \partial_A X^\mu(\sigma,\tau)(\psi_{(k)}\gamma^A(\tau^i)_{(k)(p)}\psi_{(p)})(\sigma,\tau) \right\}. \tag{C.8}$$

Note that $\{\psi_{(k)}, \bar{\psi}_{(k)}\}$ must belong to the fundamental representation of the Chosen Flavor Group with generators $\{\tau^i\}$.

They are the vertex (and theirs Lorentz-Flavor tensor-like generalizations) which are expected to generate the meson spectrum. One remark now to be pointed out is that these vertexs are coupled to *unphysical fields* $\{J^i(x);\ A_\mu^i\tau_i\}$ *in the External Space-Time*.

The Coupling of the theory with the physical Electro-Weak Fields should be made only at the Boundary (after evaluating the random surface Path Integral equation (8) with external vertexs equations (C.7)–(C.8) which by its turn, is a functional of the loop $\{c_\mu(s)\}$); by means of the Boundary Interactions Phase Factors:

$$\mathbb{P}_{\text{flavor}}\left\{\exp - \int_0^{2\pi} d\sigma(\sigma^i(c_\mu(\sigma))\tau_i)\sqrt{|c'_\mu(\sigma)|^2}\right\}, \tag{C.9}$$

$$\mathbb{P}_{\text{flavor}}\left\{\exp\left(-i\int_0^{2\pi} d\sigma(W_\mu^i(c_\alpha(\sigma))\tau_i)c'_\mu(\sigma)\right)\right\} \tag{C.10}$$

which are obtained after reformulating the Electro-Weak Sector in the Quark-Lepton loop Space. Here $\sigma^i(x)\tau_i$ and $A_\mu^i(x)\tau_i$ are physical flavored fields.

Scattering Amplitudes (S-Matrix) of the Mesons-Leptons-Higgs-Flavor Gauge fields are, thus, in principle evaluated in the Loop Space in continuum or in a latticized approach.

In order to consider Baryons on this Geometrical Functional Integral Approach, *one must build External Fermion Vertex* and add to those (C.7)–(C.8), which still to be an open problem in this approach although there is a obscure propose by Migdal which considers directly in Loop Space a new kind of QCD Baryon Loop Wave Equation (A.M. Migdal, Phys. Rep. 102, 199 (1983)) and [3].

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